Proceedings of 24th Conference
on Decision and Control
Ft. Lauderdale, FL · December 1985

WP10 - 2:30

SECOND ORDER NECESSARY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS WITH CONTROL AND INTEGRAL CONSTRAINTS[1]

M. A. Kazemi-Dehkordi + and Kang G. Shin ++

+Department of Mathematics,
University of North Carolina at Charlotte,
Charlotte, North Carolina.

++Department of Electrical Engineering and Computer Science,
The University of Michigan,
Ann Arbor, MI 48109

ABSTRACT

An optimal control problem containing restrictions on control variables and integral equality and inequality constraints is considered. A relatively simple and self-contained method for deriving new second order necessary conditions for such problems is presented. The new conditions derived here generalize some of the results reported in [3], [6], [9]. The method developed is quite general and is also applicable to optimal control problems including state or mixed state-control inequality constraints and to optimal control problems governed by partial differential equations.

1. INTRODUCTION

In a recent paper, Warga [1] presented a simple and self-contained derivation of new second order necessary conditions for an abstract optimization problem containing restrictions in the form of finitely many equality constraints and in the form of (possibly infinite-dimensional) inclusions in closed convex sets. His elegant proof is based on the separation of convex sets and on a suitable fixed point theorem. Unlike somewhat similar necessary conditions obtained by Bernstein [2] and Bernstein and Gilbert [3], the Lagrange-type multipliers l of Warga [1] may be common to all elements of a certain set $Y_{\alpha\beta}$ of critical variations (see Section 4 below) and need not be chosen separately for each critical variation. Furthermore, the set C in [1] defining the infinite dimensional constraints may be an arbitrary closed convex set and not necessarily a cone. These appear to be important improvements over the results reported in [2], [3], [4]. See [5] for a survey of higher order necessary conditions in unrestricted optimal control problems. For related work (with different approaches) on restricted problems, see [2], [3], [4], [6], [7], [8] and references therein.

The main goal of this paper is to present a self-contained and relatively simple approach to second order conditions in optimal control problems (governed by ordinary differential equations) including constraints on the control variables, and integral equality and inequality constraints. The second order conditions that we obtain here generalize Theorem 3.1 in [3] and also the results in [6]. The theory developed can also be extended with little modification to relaxed controls (although we do not consider relaxed controls here) and, therefore, generalize some of the results of [3] and [4]. In a forthcoming paper we shall apply the basic ideas we develop here to optimal control problems with state or mixed state-control inequality constraints. The approach makes use of the dependence of solutions of differential equations on parameters, the ideas in Warga [1] and our previous work in [9].

The paper is organized as follows. In Section 2, the optimal control problem is stated. Section 3 introduces the notation and assumptions to be used in the subsequent sections. We present the main results in Section 4. Section 5 deals with the dependence of solutions of differential equations on parameters. In Section 6 proofs are completed, and the paper concludes with Section 7.

2. STATEMENT OF THE PROBLEM

Let $I := [t_1, t_2]$ be a fixed, closed, and bounded interval of the real numbers R and $S \subseteq R^r$. Let $M(I,S)$, $L_\infty(I,S)$, and $AC(I,S)$ denote, respectively, the linear spaces of all functions $F : I \to S$ which are measurable, measurable and essentially bounded, and absolutely continuous. For open sets $X \subseteq R^n$ and $U \subseteq R^m$, define

$$\mathbf{X} := AC(I,X), \quad \mathbf{U} \subset M(I,U),$$

$$f : I \times X \times U \to R^n,$$

$$\phi_i, \psi_j : I \times X \times U \to R, \quad i = 0, 1, \ldots, M; \; j = 1, \ldots, N,$$

$$\phi := (\phi_1, \ldots, \phi_M), \quad \psi := (\psi_1, \ldots, \psi_N).$$

$$\Phi := (\phi_0, \phi_1, \ldots, \phi_M),$$

$$\Psi := (\phi_0, \phi_1, \ldots, \phi_M, \psi_1, \ldots, \psi_N).$$

The optimal control problem is to find a pair $(x(\cdot), u(\cdot))$ in $\mathbf{X} \times \mathbf{U}$ which minimizes the functional

$$J(x,u) := \int_{t_1}^{t_2} \phi_0(t, x(t), u(t))\, dt \tag{1}$$

subject to

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \quad \text{a.a. } t \in I, \quad x(t_1) = x_1, \tag{2}$$

$$\int_{t_1}^{t_2} \phi_i(t, x(t), u(t))\, dt = 0, \quad i = 1, \ldots, M, \tag{3}$$

$$\int_{t_1}^{t_2} \psi_j(t, x(t), u(t))\, dt \le 0, \quad j = 1, \ldots, N. \tag{4}$$

3. NOTATION AND ASSUMPTIONS

It is necessary to introduce the following notation and assumptions concerning the functions defining the optimal control problem.

Let A be an open subset of R^r. The first derivative of a C^1 map $F : A \to R^s$ at $\bar{a} \in A$ is defined to be the linear map

$$F'(\bar{a}) : R^r \to R^s, \qquad y = F'(\bar{a})x,$$

[1]The work reported in this paper was supported in part by the NSF Grant No. ECS-8409938.

CH2245-9/85/0000-0609 $1.00 © 1985 IEEE

where $F'(\bar{a})$ is identified with the Jacobian of F at $\bar{a}$. If F is C^2 on A, then the second derivative of F at $\bar{a}$ is defined to be the bilinear map

$$F''(\bar{a}) : R^r \times R^r \to R^s, \qquad y = x_1^T F''(\bar{a}) x_2.$$

Here T denotes the transpose and $F''(\bar{a})$ is the Hessian matrix of F. The i^{th} component of the vector y is understood as $y_i = x_1^T F_i''(\bar{a}) x_2$, where $F_i''(\bar{a})$ is the $r \times r$ symmetric matrix of second partial derivatives of the i^{th} component of F at $\bar{a}$. For notational simplicity,

$$F''(\bar{a})(x_1,x_2) := x_1^T F''(\bar{a}) x_2, \; F''(\bar{a}) x^2 := F''(\bar{a})(x,x).$$

Now suppose that $F : A_1 \times A_2 \to R^s$ is C^1, where $A_i \subseteq R^{r_i}$, $i = 1,2$ are open, and let $\bar{a} = (\bar{a}_1, \bar{a}_2)$, $\bar{a}_i \in A_i$. The first partial derivative of F with respect to a_1 at $\bar{a}$, denoted by $F_{a_1}(\bar{a})$, is the first derivative of the map $F : A_1 \to R^s$, where $F(a_1) := F(a_1, \bar{a}_2)$. $F_{a_2}(\bar{a})$ is defined in the obvious manner. If F is C^2, the second partial derivative,

$$F_{a_j a_k}(\bar{a}) : R^{r_j} \times R^{r_k} \to R^s$$

is a bilinear map identified with s $r_j \times r_k$ matrices of the second partial derivatives of F_i.

<u>**Assumptions**</u>

A1. For each $t \in I$, the functions

$$(x,u) \to f(t,x,u) : X \times U \to R^n,$$
$$(x,u) \to \Phi(t,x,u) : X \times U \to R^{1+M},$$
$$(x,u) \to \psi(t,x,u) : X \times U \to R^N$$

are all C^2.

A2. For each $(x,u) \in X \times U$, the functions

$$t \to f(t,x,u) : I \to R^n,$$
$$t \to \Phi(t,x,u) : I \to R^{1+M},$$
$$t \to \psi(t,x,u) : I \to R^N$$

are all measurable.

A3. There exists a nonnegative integrable function $k : I \to R$ such that for all $(t,x,u) \in I \times X \times U$,

$$|f| + |f_x| + |f_u| + f_{xx}| + |f_{xu}| + |f_{uu}| \le k,$$

$$|\Phi| + |\Phi_x| + |\Phi_u| + |\Phi_{xx}| + |\Phi_{xu}| + |\Phi_{uu}| \le k,$$

$$|\psi| + |\psi_x| + |\psi_u| + |\psi_{xx}| + |\psi_{xu}| + |\psi_{uu}| \le k.$$

A4. U is convex in R^m.

Throughout this paper, $(\bar{x}(\cdot),\bar{u}(\cdot)) \in \mathrm{X} \times \mathrm{U}$ denotes an optimal pair, i.e., a solution to the optimal control problem (1) - (4). The evaluation of functions on $(\bar{x}(\cdot),\bar{u}(\cdot))$ will be represented by a superbar, e.g., $\bar{f}_x(t) := f_x(t,\bar{x}(t),\bar{u}(t))$.

4. MAIN RESULTS

In this section we state main results of the paper and give the proofs in the following sections.

We begin by defining the sets $\tilde{\mathrm{X}}$ and $\tilde{\mathrm{U}}$ of, respectively, state and control variations.

(i) $\tilde{\mathrm{X}} := AC(I, R^n)$,

(ii) $\tilde{\mathrm{U}} \subseteq L_\infty(I, R^m)$,

(iii) $\tilde{\mathrm{U}} \subseteq \mathrm{U} - \bar{u}(\cdot) := \left\{ v(\cdot) \,\middle|\, v(\cdot) = u(\cdot) - \bar{u}(\cdot),\ u(\cdot) \in \mathrm{U} \right\}$,

(iv) $0 \in \tilde{\mathrm{U}}$.

The set of critical variations is defined as one of the following two sets

$$Y_\alpha := \left\{ (y(\cdot),v(\cdot)) \in \tilde{\mathrm{X}} \times \tilde{\mathrm{U}} \,\middle|\, (y(\cdot),v(\cdot)) \text{ satisfies } (\alpha) \right\}, \quad (5)$$

$$Y_{\alpha\beta} := \left\{ (y(\cdot),v(\cdot)) \in \tilde{\mathrm{X}} \times \tilde{\mathrm{U}} \,\middle|\, (y(\cdot),v(\cdot)) \text{ satisfies both } (\alpha) \text{ and } (\beta) \right\}, \quad (6)$$

where

$$(\alpha) \quad \int_{t_1}^{t_2} \left[\bar{\phi}_{0x}(t) y(t) + \bar{\phi}_{0u}(t) v(t) \right] dt \le 0,$$

$$\int_{t_1}^{t_2} \left[\bar{\phi}_x(t) y(t) + \bar{\phi}_u(t) v(t) \right] dt = 0,$$

$$\int_{t_1}^{t_2} \left[\bar{\psi}_{jx}(t) y(t) + \bar{\psi}_{ju}(t) v(t) \right] dt \le 0, \; j = 1, \ldots, N; \quad (7)$$

$$\frac{dy(t)}{dt} = \bar{f}_x(t) y(t) + \bar{f}_u(t) v(t), \text{ a.a. } t \in I, \; y(t_1) = 0.$$

(β) For $(y(\cdot),v(\cdot))$ and $(\eta(\cdot),w(\cdot))$ in Y_α with $v(\cdot) \ne w(\cdot)$,

$$\int_{t_1}^{t_2} \left[\bar{\phi}_{0xx}(t)(y(t),\eta(t)) + \bar{\phi}_{0uu}(t)(v(t),w(t)) \right] dt \le 0,$$

$$\int_{t_1}^{t_2} \left[\bar{\phi}_{xx}(t)(y(t),\eta(t)) + \bar{\phi}_{uu}(t)(v(t),w(t)) \right] dt = 0, \quad (8)$$

$$\int_{t_1}^{t_2} \left[\bar{\psi}_{jxx}(t)(y(t),\eta(t)) + \bar{\psi}_{juu}(t)(v(t),w(t)) \right] dt \le 0,$$

$$j = 1, \ldots, N.$$

Remark 4.1. Recall that for unrestricted problems, that is for $M = 0$, $N = 0$, an optimal control $\bar{u}(t)$ is said to be ***singular*** in the sense of Pontryagin's maximum principle, if there is a nonempty subset $\tilde{\mathrm{U}}$ of $\mathrm{U} - \bar{u}(\cdot)$ (with U convex) such that (see [5],[9])

$$H_u(t,\bar{x}(t),\bar{u}(t),\lambda(t)) v(t) = 0 \text{ for all } v(\cdot) \in \tilde{\mathrm{U}}. \quad (9)$$

Here $H = \phi_0 + \lambda^T f$, $d\lambda(t)/dt = -H_x(t,\bar{x}(t),\bar{u}(t),\lambda(t))$, and $\lambda(t_2) = 0$. Condition (9) is equivalent to saying that $\int_{t_1}^{t_2} \phi_{0x}(t,\bar{x}(t),\bar{u}(t)) y(t) dt = 0$, where

$y(t)$ is the state variation defined in (7). Thus the above definition of critical variation can be regarded as an extension of this definition of singular controls to the constrained optimal control problems.

Remark 4.2. Condition (8) is essential in proving satisfactory second order optimality conditions in *restricted* optimization problems if it is desired to have a Lagrange-type multiplier l independent of critical variations (that is, l common to all elements of the set $Y_{\alpha\beta}$). See [1]. However, in the absence of (8), one can derive somewhat less satisfactory conditions in which the multipliers l have to be chosen separately for each critical variation (see Theorem 4.2 below and [3]).

In what follows, the following definitions will be used.

$$l := (l_0, l_1, l_2) := (l_0, l_1, l_{21}, \ldots, l_{2N}) \in R^{1+M+N},$$

$$\begin{aligned} H(t,x,u,\lambda) &:= l^T \Psi + \lambda^T f \; : \\ &= l_0 \phi_0 + l_1^T \phi + l_2^T \psi + \lambda^T f \end{aligned} \tag{10}$$

$$J_a := \left\{ j \in \{l, \ldots, N\} \;\middle|\; \int_{t_1}^{t_2} \bar{\psi}_j(t)\, dt = 0 \right\}$$

We now state our main results.

Theorem 4.1. If $(\bar{x}(\cdot), \bar{u}(\cdot))$ solves the problem (1) - (4) under the assumptions A1 - A4, then there exist $l \in R^{1+M+N}$ and $\lambda(\cdot) \in \widetilde{X}$, such that

$$l \neq 0, \;\; l_0 \geq 0, \tag{11}$$

$$l_{2j} \geq 0, \;\; j \in \{1, \ldots, N\}, \tag{12}$$

$$l_{2j} = 0, \; j \in \{1, \ldots, N\} \setminus J_a, \tag{13}$$

$$\lambda(t) = \int_t^{t_2} \left[l^T \bar{\Psi}_x(t) + \lambda^T(t) \bar{f}_x(t) \right] dt, \tag{14}$$

$$\int_{t_1}^{t_2} \bar{H}_u(t,\lambda(t)) v(t)\, dt \geq 0, \text{ for all } v(\cdot) \in \widetilde{U}, \tag{15}$$

$$\begin{aligned} \int_{t_1}^{t_2} \Big[&\bar{H}_{xx}(t,\lambda(t))(y(t))^2 + 2\bar{H}_{xu}(t,\lambda(t))(y(t),v(t)) \\ &+ \bar{H}_{uu}(t,\lambda(t))(v(t))^2 \Big] dt \geq 0, \text{ for all } (y(t), v(\cdot)) \in Y_{\alpha\beta}. \end{aligned} \tag{16}$$

Remark 4.3. Note that condition (16) holds for all elements $(y(\cdot), v(\cdot)) \in Y_{\alpha\beta}$. That is, the theorem asserts that a Lagrange-type multiplier l and the corresponding λ exist such that (16) is satisfied for all the elements of $Y_{\alpha\beta}$.

Theorem 4.2. Under the same conditions of Theorem 4.1, for each $(y(\cdot), v(\cdot)) \in Y_\alpha$ there exist $l \in R^{1+M+N}$ and $\lambda(\cdot) \in \widetilde{X}$ such that (11) - (16) are satisfied and

$$\begin{aligned} \int_{t_1}^{t_2} \Big[&\bar{H}_{xx}(t,\lambda(t))(y(t))^2 + 2\bar{H}_{xu}(t,\lambda(t))(y(t),v(t)) \\ &+ \bar{H}_{uu}(t,\lambda(t))(v(t))^2 \Big] dt \geq 0. \end{aligned} \tag{17}$$

Remark 4.4. Theorem 4.1 corresponds to Theorem A in [1] for abstract optimization problems. It generalizes Theorem 3.1 in [3] and thus the result in [6]. Theorem 4.2 is less satisfactory in the sense that the second order optimality condition (16) may not hold for all $(y(\cdot), v(\cdot)) \in Y_\alpha$. In other words, the Lagrange-type l may not be common to all the elements of the set of critical variations. We will prove both Theorems 4.1 and 4.2 by the same argument (although the proof is much simpler for Theorem 4.2).

For additional remarks and elaborations on Theorem 4.1, we refer the reader to [3].

5. PARAMETER DEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

In this section we consider a suitable perturbation of the differential equation (2) and study the dependence of the solution of the perturbed equation on the parameters defining the perturbation. Materials in this section are a basis to the proof of Theorems 4.1 and 4.2.

Let ϵ, $|\epsilon| < 1$, be a small parameter and let $v(\cdot), w(\cdot) \in \widetilde{U}$ be bounded measurable functions. Clearly, for sufficiently small ϵ, the function $u_\epsilon := \bar{u} + \epsilon v + \frac{1}{2}\epsilon^2 w$ belongs to the open set U. Consider now the initial value problem:

$$\frac{dx_\epsilon(t)}{dt} = f(t, x_\epsilon(t), u_\epsilon(t)), \text{ a.a. } t \in I, \; x_\epsilon(t_1) = x_1. \tag{18}$$

For $\epsilon = 0$ we have $x_\epsilon(t) = \bar{x}(t)$, a.a. $t \in I$ by the uniqueness of solutions of ordinary differential equations. Let $y(\cdot)$ be the solution of the linear initial-value problem.

$$Ly(t) = \bar{f}_u(t) v(t), \;\; y(t_1) = 0. \tag{19}$$

where $$Ly := \frac{dy}{dt} - \bar{f}_x y.$$

The derivative $\partial x_\epsilon(t) / \partial \epsilon$ exists (see, e.g., [10], Theorem II.4.9, p. 195) and for $\epsilon = 0$ we have

$$\left(\frac{\partial x_\epsilon(t)}{\partial \epsilon} \right)_{\epsilon=0} = y(t), \;\; a.a. t \in I$$

The second derivative of $x_\epsilon(t)$ at $\epsilon = 0$ exists by the same argument and thus setting

$$z(t) := \left(\frac{\partial^2 x_\epsilon(t)}{\partial \epsilon^2} \right)_{\epsilon=0}$$

we have for $a.a. t \in I$,

$$\begin{aligned} Lz(t) = &\bar{f}_u(t) w(t) + \bar{f}_{xx}(t)(y(t))^2 \\ &+ 2\bar{f}_{xu}(t)(y(t), v(t)) + \bar{f}_{uu}(t)(v(t))^2, \end{aligned} \tag{20}$$

$$z(t_1) = 0.$$

Here the linear operator L and the function y are as defined in (19).

Next let $\epsilon \geq 0$, $c_{ij} \geq 0$, and $d_{ir} \geq 0$ be nonnegative real numbers satisfying

$$\sum_{j=1}^{k} c_{ij} = 1 \text{ for each } i = 0, 1, \ldots, M,$$

$$\sum_{r=1}^{l} d_{ir} = 1 \text{ for each } i = 0,1, \ldots, M,$$

$$\epsilon \sum_{j=1}^{k} \left[\sum_{i=0}^{M} c_{ij} \right]^{\frac{1}{2}} + \frac{1}{2} \epsilon^2 \sum_{r=1}^{l} \left[\sum_{i=0}^{M} d_{ir} \right] \leq 1,$$

where k and l are arbitrary positive integers. Also let

$$\theta := (\theta_0, \theta_1, \ldots, \theta_M), \quad \theta_i \geq 0, \quad |\theta| := \sum_{i=0}^{M} \theta_i = 1.$$

Finally, define

$$u_\epsilon(t) := \bar{u}(t) + \epsilon v(t) + \frac{1}{2}\epsilon^2 w(t)$$

$$:= \bar{u}(t) + \epsilon \sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} v_j(t) \tag{21}$$

$$+ \frac{1}{2}\epsilon^2 \sum_{r=1}^{l} \left[\sum_{i=0}^{M} d_{ir} \theta_i \right] w_r(t),$$

where $v_j(\cdot)$ and $w_j(\cdot)$ belong to $\widetilde{\mathrm{U}}$. Obviously, for $0 \leq \epsilon < 1$, $u_\epsilon(\cdot)$ takes values in the set U. Consider now the initial-value problem (18) with $u_\epsilon(t)$ defined in (21):

$$\frac{dx_\epsilon(t)}{dt} = f(t, x_\epsilon(t), u_\epsilon(t)), \quad \text{a.a. } t \in I, \quad x_\epsilon(t_1) = x_1 \tag{22}$$

Lemma 5.1. The solution $x_\epsilon(\cdot)$ of the initial-value problem (22) is of the form

$$x_\epsilon(t) = \bar{x}(t) + \epsilon \sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} y_j(t)$$
$$+ \frac{1}{2} \epsilon^2 \sum_{r=1}^{l} \left[\sum_{i=0}^{M} \theta_i d_{ir} \right] z_r(t) + o(\epsilon^2),$$

where $o(\epsilon^2) \to 0$ uniformly in t as $\epsilon \to 0+$ and $y_j(\cdot)$ and $z_r(\cdot)$ are solutions of

$$Ly_j(t) = \bar{f}_u(t) v_j(t), \text{ a.a. } t \in I, \quad y_j(t_1) = 0, \; j = 1, \ldots, k, \tag{23}$$

$$Lz_r(t) = \bar{f}_u(t) w_r(t) + \bar{f}_{xx}(t)(y(t))^2 + 2\bar{f}_{xu}(t)(y(t), v(t)) + \bar{f}_{uu}(t)(v(t)),$$

$$\text{a.a. } t \in I, \quad z_r(t_1) = 0, \quad r = 1, \ldots, l,$$

where

$$y(t) := \sum_{j=1}^{k} \left[\sum_{i-0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} y_j(t), \quad \text{a.a. } t \in I. \tag{24}$$

Proof. Since the first and second derivatives of x_ϵ with respect to ϵ exist, we can write

$$x_\epsilon(t) := \bar{x}(t) + \epsilon y(t) + \frac{1}{2}\epsilon^2 z(t) + o(\epsilon^2).$$

$$Lz = \bar{f}_u w + \bar{f}_{xx} y^2 + 2\bar{f}_{xu}(y, v) + \bar{f}_{uu} v^2, \tag{26}$$

$z(t_1) = 0$, w as defined in (21).

We have to specify the forms of $y(\cdot)$ and $z(\cdot)$ in these relations. It follows from (23) that

$$L \sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} y_j(t) = \bar{f}_u(t) \sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} v_j(t), \text{ a.a. } t \in I, \tag{27}$$

$$\sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} y_j(t_1) = 0.$$

Comparing (23) and (27) and taking into account the uniqueness of solution, we see that

$$y(t) = \sum_{j=1}^{k} \left[\sum_{i=0}^{M} \theta_i c_{ij} \right]^{\frac{1}{2}} y_j(t).$$

Similarly, from (24) and definitions of c_{ij} and d_{ir} we have

$$L \sum_{r=1}^{l} \left[\sum_{i=0}^{M} \theta_i d_{ir} \right] z_r(t) = \bar{f}_u(t) \sum_{r=1}^{l} \left[\sum_{i=0}^{M} \theta_i d_{ir} \right] w_r(t) + \bar{f}_{xx}(t)(y(t))^2$$
$$+ 2\bar{f}_{xu}(t)(y(t), v(t)) + \bar{f}_{uu}(t)(v(t))^2, \tag{28}$$
$$\text{a.a. } t \in I, \quad \sum_{r=1}^{l} \left[\sum_{i=0}^{M} \theta_i d_{ir} \right] z_r(t_1) = 0.$$

Comparing (28) and (26), we see by the uniqueness of solution that z has the form asserted in the Lemma. This completes the proof.

Lemma 5.2.:

$$\int_{t_1}^{t_2} \Phi(t, x_\epsilon(t), u_\epsilon(t)) dt = \int_{t_1}^{t_2} \bar{\Phi}(t) dt + \frac{1}{2}\epsilon^2 H\theta + \alpha(\epsilon^2, \theta) + (\tilde{a}_0, 0)$$

where $\alpha(\epsilon^2, \theta) \to 0+$ uniformly in t as $\epsilon \to 0+$, H is an $(M+1) \times (M+1)$ matrix whose columns are defined by the $(M+1)$ vectors

$$\int_{t_1}^{t_2} \left[\bar{\Phi}_{xx}(t) \sum_{j=1}^{k} c_{ij}(y_j(t))^2 + 2\bar{\Phi}_{xu}(t) \sum_{j=1}^{k} c_{ij}(y_j(t), v_j(t)) \right.$$
$$+ \bar{\Phi}_{uu}(t) \sum_{j=1}^{k} c_{ij}(v_j(t))^2 + \bar{\Phi}_x(t) \sum_{r=1}^{l} d_{ir} z_r(t)$$
$$\left. + \bar{\Phi}_u(t) \sum_{r=1}^{l} d_{ir} w_r(t) \right] dt,$$

and where $\tilde{a}_0$ is the nonpositive number

$$\tilde{a}_0 := \epsilon \int_{t_1}^{t_2} \Bigg[\bar{\phi}_{ox}(t) \sum_{j=1}^{k} \left(\sum_{i=0}^{M} \theta_i c_{ij} \right)^{\frac{1}{2}} y_j(t) + \bar{\phi}_{ou}(t) \sum_{j=1}^{k} \left(\sum_{i=0}^{M} \theta_i c_{ij} \right)^{\frac{1}{2}} v_j(t) \Bigg] dt + \frac{1}{2}\epsilon^2 \int_{t_1}^{t_2} \Bigg[\bar{\phi}_{oxx}(t) \sum_{\substack{p,j=1\\ p\neq j}}^{k} \left(\sum_{i=0}^{M} \theta_i c_{ij} \right)^{\frac{1}{2}} \left(\sum_{i=0}^{M} \theta_i c_{ip} \right)^{\frac{1}{2}} (y_j(t), y_p(t))$$

$$+ \bar{\phi}_{ouu}(t) \sum_{\substack{p,j=1\\ p\neq j}}^{k} \left(\sum_{i=0}^{M} \theta_i c_{ij} \right)^{\frac{1}{2}} \left(\sum_{i=0}^{M} \theta_i c_{ip} \right)^{\frac{1}{2}} (v_j(t), v_p(t)) \Bigg] dt.$$

Proof. Write the first three terms of the Taylor series expansion of Φ about $\epsilon = 0$, use Lemma 5.1 and the fact that $(y_j(\cdot), v_j(\cdot)) \in Y_{\alpha\beta}$.

6. COMPLETION OF PROOFS

We shall use the following lemma on separation of convex sets, which is a special case of Lemma V.2.1, p.299 of [10].

Lemma 6.1. Let C be a convex subset of $R \times R^M \times R^N$, K an open convex subset of R^N, $0 \in C$, $0 \in \bar{K}$. Then either there exists $l := (l_0, l_1, l_2) \in [0,\infty) \times R^M \times R^N$ such that

$$l \neq 0,$$

$$lc := l_0 c_0 + l_1 c_1 + l_2 c_2 \geq 0 \quad \forall \ c := (c_0, c_1, c_2) \in C,$$

$$l_2 k \leq 0 \quad \forall \quad k \in \bar{K},$$

or there exist points $\xi^i := (\xi_0^i, \xi_1^i, \xi_2^i) \in C$ and numbers $\beta_i > 0$, $i = 0,1,\ldots,M$ such that $\sum_{i=0}^{M} \beta_i = 1$, $\xi_2^i \in K$, the set $\left\{ (\xi_0^0, \xi_1^0), \ldots, (\xi_0^M, \xi_1^M) \right\}$ is linearly independent, $\xi_0^i < 0$, $i = 0,1,\ldots,M$, and $\sum_{i=0}^{M} \beta_i \xi_1^i = 0$.

Proof of Theorem 4.1. Define the sets

$$K := \left\{ k \in R^N \,\middle|\, k_j < 0 \text{ for } j \in J_a \right\}$$

$$C := co \Bigg\{ \int_{t_1}^{t_2} \Big[\bar{\Psi}_{xx}(t)(y(t))^2 + 2\bar{\Psi}_{xu}(t)(y(t), v(t)) + \bar{\Psi}_{uu}(t)(v(t))^2 + \bar{\Psi}_x(t) z(t) + \bar{\Psi}(t) w(t) \Big] dt \in R^{1+M+N} \,\Bigg|\, (y(\cdot), v(\cdot)) \in Y_{\alpha\beta},\ (z(\cdot), w(\cdot)) \text{ satisfy } (20) \Bigg\}$$

$$l \neq 0, \ l_0 \geq 0, \ l_2 k \leq 0 \quad \forall \ k \in \bar{K} \tag{29}$$

$$l^T \int_{t_1}^{t_2} \Big[\bar{\Psi}_{xx}(t)(y(t))^2 + 2\bar{\Psi}_{xu}(t)(y(t), v(t)) + \bar{\Psi}_{uu}(t)(v(t))^2 + \bar{\Psi}_x(t) z(t) + \bar{\Psi}_u(t) w(t) \Big] dt \geq 0 \tag{30}$$

where co denotes the convex hull. Clearly, K and C are convex, $0 \in \bar{K}$, $0 \in C$ and K is open. Thus by Lemma 6.1, either there exists $l := (l_0, l_2, l_2) \in [0,\infty) \times R^M \times R^N$ satisfying

Remark 5.1. Similar expansion holds for $\psi := (\psi_1, \ldots, \psi_N)$.

$$l \neq 0, \ lc \geq 0, \ \forall \ c \in C, \text{ and } l_2 k \leq 0 \quad \forall \ k \in \bar{K}.$$

Or there exist $\xi^i := (\xi_0^i, \xi_1^i, \xi_1^i) \in C$ and β_i, $i = 0,1,\ldots,M$ such that the set $\left\{ (\xi_0^i, \xi_1^i) \,\middle|\, i = 0,1,\ldots,M \right\}$ is linearly independent, $\xi_0^i < 0$, $\xi_2^i \in K$, $i = 0,1,\ldots,M$, $\beta_i > 0$, $i = 0,1,\ldots,M$, $\sum_{i=0}^{M} \beta_i = 1$ and $\sum_{i=0}^{M} \beta_i \xi_1^i = 0$.

From the first alternative we obtain

for all $(y(\cdot), v(\cdot)) \in Y_{\alpha\beta}$ and all $(z(\cdot), w(\cdot))$ satisfying (20).

We prove that the second alternative cannot hold for the above choices of K and C. So assume, on the contrary, that the second alternative holds. Thus there are nonnegative numbers c_{ij}, $\sum_{j=1}^{k_i} c_{ij} = 1$ for $i = 0,1,\ldots,M$ and $(y_{ij}(\cdot), v_{ij}(\cdot))$ in $Y_{\alpha\beta}$, $(z_{ij}(\cdot), w_{ij}(\cdot))$ satisfying (21) such that

$$\xi^i := (\xi_0^i, \xi_1^i, \xi_2^i) = \sum_{j=1}^{k_i} c_{ij} \int_{t_1}^{t_2} \Big[\bar{\Psi}_{xx}(t)(y_{ij}(t))^2 + 2\bar{\Psi}_{xu}(t)(y_{ij}(t), v_{ij}(t)) + \bar{\Psi}_{uu}(t)(v_{ij}(t))^2 + \bar{\Psi}_x(t) x_{ij}(t) + \bar{\Psi}_u(t) w_{ij}(t) \Big] dt.$$

Representing all the distinct $(y_{ij}(\cdot), v_{ij}(\cdot))$ as $(y_1(\cdot), v_1(\cdot)), \ldots, (y_k(\cdot), v_k(\cdot))$, all the distinct $(z_{ij}(\cdot), w_{ij}(\cdot))$ as $(z_1(\cdot), w_1(\cdot)), \ldots, (z_1(\cdot), w_1(\cdot))$ and redefining c_{ij} appropriately, we can write

$$\xi^i := (\xi_0^i, \xi_1^i, \xi_2^i) = \Bigg\{ \sum_{j=1}^{k} c_{ij} \int_{t_1}^{t_2} \Big[\bar{\Phi}_{xx}(t)(y_j(t))^2 + 2\bar{\Phi}_{xu}(t)(y_j(t), v_j(t)) + \bar{\Phi}_{uu}(t)(v_j(t))^2 \Big] dt + \sum_{r=1}^{l} d_{ir} \int_{t_1}^{t_2} \Big[\bar{\Phi}_x(t) z_r(t) + \bar{\Phi}_u(t) w_r(t) \Big] dt,\ \sum_{j=1}^{k} c_{ij} \int_{t_1}^{t_2} \Big[\bar{\psi}_{xx}(t)(y_j(t))^2$$

$$+ 2\bar{\psi}_{xu}(t)(y_j(t), v_j(t) + \bar{\psi}_{uu}(t)(v_j(t))^2 \Big] dt + \sum_{r=1}^{l} d_{ir} \int_{t_1}^{t_2} \Big[\bar{\psi}_x(t) z_r(t) + \bar{\psi}_u(t) w_r(t) \Big] dt \Bigg\}.$$

Using c_{ij} and d_{ir}, the functions $v_j(\cdot)$ and $w_r(\cdot)$ in (21) we define $u_\epsilon(\cdot)$ as in (20) and find the corresponding $x_\epsilon(t)$ in (21). By Lemma 5.2 we have

$$\int_{t_1}^{t_2} \overline{\Phi}(t, x_\epsilon(t), u_\epsilon(t))\,dt = \int_{t_1}^{t_2} \overline{\Phi}(t)\,dt + \frac{1}{2}\epsilon^2 H\theta + \alpha(\epsilon^2,\theta) + (\tilde{a}_0, 0), \tag{32}$$

where now the $(1+M)\times(1+M)$ matrix H with linearly independent columns (ξ_0^i, ξ_1^i) defined in (31), $i = 0,1,\ldots,M$, is nonsingular. Let

$$\beta_{\min} := \min\left\{\beta_0, \beta_1, \ldots, \beta_M\right\}$$

and observe that there exists $\overline{\epsilon} > 0$ such that

$$|2H^{-1}\alpha(\epsilon^2,\theta)| \le \frac{\epsilon}{3}\beta_{\min} \text{ for } |\theta| = 1 \text{ and } 0 \le \epsilon < \overline{\epsilon} \tag{33}$$

Choose $\overline{\epsilon}$ sufficiently small so that

$$\overline{\epsilon}\sum_{j=1}^{k}\left(\sum_{i=0}^{M} c_{ij}\right)^{\frac{1}{2}} + \frac{1}{2}\overline{\epsilon}^2\sum_{r=1}^{l}\left|\sum_{i=0}^{M} d_{ir}\right| \le 1,$$

and

$$\gamma := \frac{2}{3}\overline{\epsilon},\ S := \left\{ s = (s_0, s_1, \ldots, s_M) \in \boldsymbol{R}^{M+1} \;\middle|\; |s_i - \gamma\beta_i| \le \frac{1}{2}\gamma\beta_{\min},\ 0 \le i \le M \right\}.$$

The set S is clearly compact and convex and

$$s \in S \text{ implies } 0 < \frac{1}{2}\gamma\beta_{\min} \le s_i \text{ and } |s| \le \overline{\epsilon}. \tag{34}$$

Furthermore, the mapping

$$s \to \gamma\beta - \frac{2}{|s|}H^{-1}\alpha\left(|s|, \frac{s}{|s|}\right) : S \to \boldsymbol{R}^{1+M},$$

where $\beta := (\beta_0, \beta_1, \ldots, B_M)$ is continuous and it easily follows from (33)-(34) that it maps S into S. Therefore, it has a fixed point $\hat{s} \in S$. Let us write $\hat{s} = \hat{\epsilon}\hat{\theta}$, $\hat{\epsilon} = |\hat{s}|$, $|\hat{\theta}| = 1$. Thus we have shown that for some $\hat{\epsilon} > 0$ and $\hat{\theta}$, $|\hat{\theta}| = 1$,

$$\alpha(\hat{\epsilon}^2, \hat{\theta}) = \frac{1}{2}\hat{\epsilon}\gamma H\beta - \frac{1}{2}\hat{\epsilon}^2 H\hat{\theta}$$

and so, defining $x_{\hat{\epsilon}}(\cdot)$ and $\hat{a}_0$ the same way as $x_\epsilon(\cdot)$ and $\tilde{a}_0$ but with $\hat{\epsilon}$, $\hat{\theta}$ replacing ϵ, θ, it follows from (32) that

$$\int_{t_1}^{t_2} \Phi(t, x_{\hat{\epsilon}}(t), u_{\hat{\epsilon}}(t))\,dt = \int_{t_1}^{t_2} \overline{\Phi}(t)\,dt + \frac{1}{2}\hat{\epsilon}\gamma H\beta + (\hat{a}_0, 0),$$

from which we deduce

$$\int_{t_1}^{t_2} \phi_0(t, x_{\hat{\epsilon}}(t), u_{\hat{\epsilon}}(t))\,dt = \int_{t_1}^{t_2} \overline{\phi}_0(t)\,dt + \frac{1}{2}\hat{\epsilon}\gamma\sum_{i=0}^{M}\beta_i\xi_0^i + \hat{a}_0$$

$$< \int_{t_1}^{t_2} \overline{\phi}_0(t)\,dt + \hat{a}_0 \le \int_{t_1}^{t_2} \overline{\phi}_0(t)\,dt,$$

$$\int_{t_1}^{t_2} \phi(t, x_{\hat{\epsilon}}(t), u_{\hat{\epsilon}}(t))\,dt = \int_{t_1}^{t_2} \overline{\phi}(t)\,dt + \frac{1}{2}\hat{\epsilon}\gamma\sum_{i=0}^{M}\beta_i\xi_1^i$$

$$= \int_{t_1}^{t_2} \overline{\phi}(t)\,dt = 0.$$

We need to show that the pair $(x_{\hat{\epsilon}}(\cdot), u_{\hat{\epsilon}}(\cdot))$ satisfies the inequality constraints (4). An expression similar to the one in (32) yields

$$\int_{t_1}^{t_2} \psi(t, x_{\hat{\epsilon}}(t), u_{\hat{\epsilon}}(t))\,dt = \int_{t_1}^{t_2} \overline{\psi}(t)\,dt + \frac{1}{2}\hat{\epsilon}^2\sum_{i=0}^{M}$$

$$\theta_i\left\{\int_{t_1}^{t_2}\left[\overline{\psi}_{xx}(t)\sum_{j=1}^{k} c_{ij}(y_j(t))^2 + 2\overline{\psi}_{xu}(t)(y_j(t), v_j(t)) + \overline{\phi}_{uu}(t)(v_j(t))^2\right] dt + \int_{t_1}^{t_2}\sum_{r=1}^{l} d_{ir}\left[\overline{\psi}_x(t)z_r(t) + 2\overline{\psi}_u(t)w_r(t)\right] dt\right\} + \tilde{a}_2 + \alpha(\epsilon^2,\theta)$$

$$= \int_{t_1}^{t_2} \overline{\psi}(t)\,dt + \frac{1}{2}\epsilon^2\sum_{i=0}^{M}\theta_i\xi_2^i + \tilde{a}_2 + \alpha(\epsilon^2, 0),$$

where $\tilde{a}_2 := (\tilde{a}_{21}, \ldots, \tilde{a}_{2N})$, $\tilde{a}_{2j} \le 0$, is defined similar to $\tilde{a}_0$ but with ψ replacing ϕ_0, and where $\tilde{\alpha}(\epsilon^2,\theta) \to 0$ as $\epsilon \to 0+$ uniformly in θ. Noting that $\xi_2^i \in K$, and $a_{2j} \le 0$, $j = 1, \ldots, N$, we see that for sufficiently small ϵ (specifically $\epsilon = \hat{\epsilon}$) each component of the right-hand side of (35) is nonpositive. In conclusion, we have proven that the pair $(x_{\hat{\epsilon}}(\cdot), u_{\hat{\epsilon}}(\cdot))$ is an admissible one at which the cost functional is less than the minimum value. This contradiction shows that the second alternative cannot hold. We now derive the optimality conditions (11)-(16) from (29) and (30).

Conditions (11)-(13) are just equivalent forms of (29). To derive (14)-(16), define the absolutely continuous function λ on I by

$$\lambda(t) = \int_{t}^{t_2}\left[l^T\overline{\Psi}_x(t) + \lambda(t)^T\overline{f}_x(t)\right] dt \tag{36}$$

Setting $v = 0$ in (30) we have

$$\int_{t_1}^{t_2} l^T\left[\overline{\Psi}_x(t)z(t) + \overline{\Psi}_u(t)w(t)\right] dt \ge 0 \tag{37}$$

for all $(z(\cdot), w(\cdot))$ satisfying (20). But notice that (20) with $v = 0$ is equivalent to

$$\frac{dz(t)}{dt} = \bar{f}_z(t)z(t) + \bar{f}_u(t)w(t), \quad z(t_1) = 0. \quad \mathbf{(38)}$$

Application of integration by parts to the first term of (37) yields

$$\int_{t_1}^{t_2} l^T \bar{\Psi}_z(t)z(t)dt = \int_{t_1}^{t_2} \left[\int_t^{t_2} l^T \bar{\Psi}_z(\tau)d\tau \right] \frac{dz(t)}{dt} dt \quad (39)$$

Substituting from (36) into (39), using (38), simplifying, and finally substituting the result back into (37) yields (15).

To derive (16), we again use (30) but this time set $w = 0$, so that z now satisfies (see (20)).

$$\frac{dz(t)}{dt} = \bar{f}_z(t)z + \bar{f}_{zz}(t)(y(t))^2 + 2\bar{f}_{zu}(t)(y(t),v(t))$$
$$+ \bar{f}_{uu}(t)(v(t))^2, \quad z(t_1) = 0, \quad (40)$$

where $(y(\cdot),v(\cdot))$ has been defined in (19). Use the relation (39) with z as in (40) and repeat the above calculations to derive (16).

Proof of Theorem 4.2. The proof is almost exactly the same as that of Theorem 4.1. Here we define K as above but for each $(y(\cdot),v(\cdot))\in Y_\alpha$, the set C is defined as

$$C_{(y,v)} := \Bigg\{ \int_{t_1}^{t_2} \Big[\alpha\bar{\Psi}_{zz}(t)(y(t))^2 + 2\alpha\bar{\Psi}_{zu}(t)(y(t),v(t))$$
$$+ \alpha\bar{\Psi}_{uu}(t)(v(t))^2 + \bar{\Psi}_z(t)z(t) + \bar{\Psi}_u(t)w(t) \Big] dt$$
$$\in R^{1+M+N} \ \Big| \ 0 \le \alpha \le 1, \text{ and } (z(\cdot),w(\cdot)) \text{ satisfying } (20) \Bigg\}.$$

Clearly $C_{(y,v)}$ is convex and contains zero (taking $\alpha=0$, $w(\cdot)\equiv 0$). Accordingly instead of (21) here we define

$$u_\epsilon(t) := \bar{u}(t) + \epsilon\alpha v(t) + \frac{1}{2}\epsilon^2 \sum_{r=1}^{l} \left(\sum_{i=0}^{M} \theta_i d_{ir} \right) w_i(t), \quad t\in I,$$

and proceed as in the proof of Theorem 4.1. Note that in the present situation, the expansions of Φ and ψ_o around $\epsilon = 0$ contain no cross product terms such as, for example, $\bar{\phi}_{ozz}(t)y_i(t)y_j(t)$. The corresponding terms $\hat{a}_0$ and $\tilde{a}_2$ are still nonpositive in here and thus the proof proceeds exactly as in Theorem 4.1.

7. CONCLUSION

In this paper, we have developed a new method for deriving second order necessary conditions for optimal control problems. The method is applicable to a wide range of optimal control problems including state or mixed state-control inequality constraints. It is also suitable for control systems with distributed parameters.

REFERENCES

[1] Warga, J., "Second Order Necessary Conditions in Optimization", *SIAM Journal on Control and Optimization,* Vol. 22, pp. 524-528, 1984.

[2] Bernstein, D. S., "A systematic Approach to Higher-Order Necessary Conditions in Optimization Theory", *SIAM Journal on Control and Optimization* Vol. 22, pp. 211-238, 1984.

[3] Bernstein, D. S. and Gilbert, E. G., "Second-Order Necessary Conditions in Optimal Control: Accessory-Problem Results without Normality Conditions", *Journal of Optimization Theory and Applications,* Vol. 41, NO. 1, pp. 75-106, 1983.

[4] Warga, J., "A Second Order Condition that Strengthens Pontryagin's Maximum Principle", *Journal of Differential Equations,* Vol. 28, pp. 284-307, 1979.

[5] Gabasov, R., and Kirillov, F. M., "High Order Necessary Conditions for Optimality", *SIAM Journal on Control and Optimization,* Vol. 10, pp. 127-168, 1972.

[6] Warga, J., "A Second Order Lagrangian Condition for Restricter Control Problems", *Journal of Optimization Theory and Applications* Vol. 22, pp. 475-483, 1978.

[7] Krener, A. J., "The High Order Maximum Principle and its Application to Singular Extremals", *SIAM Journal on Control and Optimization,* Vol. 15, pp. 256-293, 1977.

[8] Agrachev, A. A., and Gamkrelidze, R. V., "A Second-Order Optimality Principle for a Time-Optimal Problem", *Mathematics USSR Sbornik,* Vol. 29, pp. 547-546, 1976.

[9] Kazemi-Dehkordi, M. A., "Necessary Conditions for Optimality of Singular Controls", *Journal of Optimization Theory and Applications,* Vol. 43, No. 4, pp. 629-637, 1984.

[10] Warga, J., *Optimal Control of Differential and Functional Equations,* Academic Press, New York, New York, 1972.